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EFFECTIVE MODELS OF COMPOSITE PERIODIC PLATES—I. ASYMPTOTIC SOLUTION

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Abstract—This paper is concerned with the statical problem of plates with oscillating material properties and rapidly varying face shapes. Both the material moduli and the geometrical data vary periodically with the same period. By using the asymptotic method of Caillerie (1982, *C. R. Acad. Sci. Paris* **294**(II), 159), the higher-order terms of the expansion for deformations and stresses are arrived at. These terms are determined by a subsequent homogenized problem. For new auxiliary functions of a local variable, appropriate basic cell problems are established. Proof that the conventional asymptotic expansion does not comprise boundary effects at the clamped edge is also given.

1. INTRODUCTION

The problem of evaluating the effective stiffnesses of regularly non-homogeneous plate structures such as reinforced plates, plates stiffened by ribs or plates weakened periodically by openings, is indissolubly bonded with the history of thin plate theory and its engineering applications. For instance, the effective bending rigidities $D^{\alpha\alpha\alpha\alpha}$ where $\alpha = 1, 2$, of plates orthogonally stiffened by ribs are usually estimated by using a plate–beam analogy in engineering papers; the remaining stiffnesses being determined with the help of Huber's formulae, cf. Huber (1914) (originally derived for reinforced concrete plates):

$$D^{1122} = \nu(D^{1111}D^{2222})^{1/2}, \quad D^{1212} = \frac{1-\nu}{4}(D^{1111}D^{2222})^{1/2}; \tag{1}$$

where ν is Poisson's ratio. The aforementioned formulae and similar formulae used in engineering practice (which can be found in the respectable textbooks on plate structures by Timoshenko and Woinowsky-Krieger, 1959; Kączkowski, 1968; Başar and Krätzig, 1985), can be helpful but do not enter into the essence of the problem. The effective stiffnesses of plates weakened periodically by circular openings can be estimated by using the advanced tools of complex analysis (cf. Grigoliuk and Fil'shtynskii, 1970; Kosmodamianskii, 1975). This method, however, cannot be generalized to include the problem for plates of complex geometry and material properties. Moreover, one should emphasize that the rational construction of the effective models for periodic plates cannot be given by the conventional methods of variational calculus, although these methods are the basic tools for constructing a mathematical description of transversely non-homogeneous plates undergoing transverse shear deformations (cf. Reissner, 1985). Even the classical method of homogenization (Bakhvalov and Panasenko, 1984; Bensoussan *et al.*, 1978; Sanchez-Palencia, 1980), which involves one small parameter ε characterizing the dimensions of the periodicity cells, is a tool too weak to formulate effective plate models. In the initial elastostatic problem of a plate with composite periodic structure, two small parameters are present: the plate thickness h and an in-plane dimension ε of the periodicity cell. Both affect the governing equations in various ways. One can simplify the problem making use of the fact that the thickness is small prior to homogenizing periodic properties. For instance, one can stipulate the Kirchhoff constraints on the original problem and then homogenize the two-dimensional equations. The formulae for effective stiffnesses which were obtained in this manner by Duvaut (1976), have a limited range of applicability. They are valid for plates composed of cells which also possess the shapes of thin plates (in Part III of this paper, additional conditions which determine the range of applicability of Duvaut's results will be reported). Instead of Kirchhoff's constraints, we can impose those of Hencky and carry out the homogenization process of the two-dimensional equations of moderately thick periodic plates (cf. Tadlaoui and Tapiero, 1988; Lewiński and Telega, 1989; see also papers by Lewiński and Telega, 1988a,b; Telega and Lewiński, 1988, concerning thin and moderately thick plates with fissures of the Signorini type). Similarly, one can homogenize Reddy's (1984) equations of moderately thick plates (cf. also Lewiński, 1986), the equations being assumed *a priori* to be energy-consistent with the displacement fields (cf. Lewiński and Telega, 1989). The results from homogenizing Kirchhoff's, Hencky's and Reddy's equations can be shown to be dissimilar. Thus, the question of finding the range of applicability of these results arises. This question will be addressed in the third part of the present paper. In the aforementioned approaches, the homogenization is preceded by the reduction of the transverse dimension. Therefore, the transverse variations of the material properties are not properly considered since the transverse characteristics are averaged and thus suppressed at an early stage in the analysis. Consequently, the basic cell problems are two-dimensional. Such an algorithm seems to be inadequate when the plate is reinforced.

For composite plates of constant thickness, a method of simultaneous reduction of all dimensions of the periodicity cells has been put forward by Caillerie (1982), i.e. the model $e \approx \varepsilon$. This method takes advantage of the asymptotic method of Gol'denveizer (1962) and Gol'denveizer and Kolos (1965), which was smoothed out later by Ciarlet and Destuynder (1979); and of the classical homogenization. Homogeneous plates of rapidly varying thickness have been independently considered by Kohn and Vogelius (1984, 1985, 1986) with the help of methods similar to those used by Caillerie (1982). In the paper by Kalamkarov *et al.* (1987), the results of Caillerie (1982, 1984) and Kohn and Vogelius (1984), have been generalized to the case of shells with rapidly varying material properties and thickness. The present paper is aimed at deriving and discussing the higher-order terms of the asymptotic expansions that model the deformations of plates with rapidly varying properties and thickness. The plates are not assumed to be symmetric with respect to its middle plane. The derivation is based upon a new variational equation. In this formulation the domain of integration is a product domain $\Omega \times \mathscr{Y}$, where Ω is the plane reference domain and $\mathscr{Y}$ a rescaled three-dimensional cell of periodicity. Variational formulations of this type have been postulated by Mindlin (1964) for materials with microstructure (see also the monograph by Woźniak, 1969), and are also used in the mixture theory (cf. Murakami and Hegemier, 1986; Toledano and Murakami, 1987). For periodic composites, similar equations have been used by Maewal (1986) and Kucharski and Lewiński (1989) to obtain Cosserat-type approximations. This non-conventional variational formulation makes the derivation fully deductive. The crucial point is that the microcoordinate $\mathbf{y} \in \mathscr{Y}$ can be treated as an independent variable from the very beginning of the derivation. In order to make this paper understandable, not only will new results be derived but also the known main terms of the solution.

Part II of this paper will be concerned with the consequences of various symmetries of the plate and of the periodicity cell. The algorithm of computing effective stiffnesses will also be given. The final, third part of this paper will deal with two-dimensional approximations of the initial three-dimensional basic cell problems.

Throughout the paper, a conventional summation convention is adopted. The indices i, j, k, l, m, n, h and t take values 1, 2, 3; Greek indices (except for ε) run over 1, 2; and the index p takes values over all integers. Moreover, the following symbols of differentiation will be used: $\partial/\partial x_j = {}_{,j}$, $\partial/\partial y_j = {}_{|j}$.

2. THREE-DIMENSIONAL FORMULATION

The subject of consideration is a plate with varying thickness which occupies a domain

$$B = \{\mathbf{x} = (x_i) | x = (x_\alpha) \in \Omega, \quad x_3^-(x) < x_3 < x_3^+(x)\};$$

where Ω is a reference plane (in the case where $x_3^- = -x_3^+$ is the middle plane of the plate);

and (x_i) are the Cartesian coordinates. The functions $x_3^{\pm}(x)$, which determine the faces $\Gamma_{\pm} = \{\mathbf{x}|x \in \Omega, x_3 = x_3^{\pm}(x)\}$, are assumed to be Z-periodic, $Z = (0, Z_1) \times (0, Z_2)$. The elastic moduli of the plate material $C_Z^{ijkl}(x, x_3)$ are assumed to be Z-periodic in x and arbitrarily, transversely non-homogeneous. The stress fields $\sigma^{ij}(\mathbf{x})$ are associated with the linear strain measures $\gamma_{kl} = \gamma_{kl}(\mathbf{u}) = \frac{1}{2}(u_{k,l}+u_{l,k})$, according to Hooke's law. The body forces are omitted. Along the boundary $\Gamma_0 = \{\mathbf{x}|x \in \gamma = \partial\Omega,\ x_3^-(x) < x_3 < x_3^+(x)\}$, the plate is clamped. The faces $\Gamma_{\pm}$ are subjected to the loadings $p_i^{Z\pm}(x, x)$ with $p_i^{Z\pm}(x, \cdot)$ being Z-periodic functions. Thus, the boundary conditions on the faces read

$$\sigma^{ij}(x, x_3^{\pm})n_j^{\pm}(x) = p_i^{Z\pm}(x, x), \tag{2}$$

where $\mathbf{n}^{\pm}(x)$ is the unit vector outwardly normal to $\Gamma_{\pm}$ at $(x, x_3^{\pm})$.

The plate considered is $\mathscr{X}$-periodic, its periodicity cell

$$\mathscr{X} = \{\mathbf{x}|0 < x_\alpha < Z_\alpha, x_3^-(x) < x_3 < x_3^+(x)\}$$

generally not being symmetric with respect to the $x_3 = 0$ plane.

The statical problem (P) of the considered plate consists of finding the statically admissible stress field σ^{ij}, viz. such that

$$\int_B \sigma^{ij} v_{i,j}\,\mathrm{d}\mathbf{x} = \int_{\Gamma_+} p_i^{Z+} v_i(x, x_3^+)\,\mathrm{d}\Gamma_+ + \int_{\Gamma_-} p_i^{Z-} v_i(x, x_3^-)\,\mathrm{d}\Gamma_- \tag{3}$$

for every $\mathbf{v}$ that vanishes on Γ_0. The stresses are interrelated with the displacement field by Hooke's law,

$$\sigma^{ij}(\mathbf{x}) = C_Z^{ijkl}(x, x_3)\gamma_{kl}(\mathbf{u}), \tag{4}$$

and $\mathbf{u}$ is assumed to be zero on Γ_0.

3. THE FAMILY OF (P_ε) PROBLEMS

The (P) problem involves three quantities: $h = \max|x_3^+ - x_3^-|$, Z_1 and Z_2, which are small in comparison with the global dimensions of Ω. The presence of small quantities enables us to apply the method of small parameters. This method deals in constructing a family (P_ε) of problems such that for a certain $\varepsilon = \varepsilon_0$, $(P_{\varepsilon_0}) = (P)$. In every problem (P_ε), the plate should have a homothetic microstructure. Thus, we substitute

$$x_3^{\pm} \to h^{\pm}\left(\frac{x}{\varepsilon}\right) = \varepsilon c^{\pm}\left(\frac{x}{\varepsilon}\right), \quad Z_\alpha \to \varepsilon Y_\alpha;$$

$$Z \to \varepsilon Y, \quad \mathscr{X} \to \varepsilon\mathscr{Y}, \quad B \to B_\varepsilon; \tag{5}$$

where $Y = (0, Y_1) \times (0, Y_2)$ and $h^{\pm}(\cdot)$, $c^{\pm}(\cdot)$ are Y-periodic functions. We also define $y_3 = x_3/\varepsilon$, cf. Gol'denveizer (1962). The rescaled cell of periodicity is defined by

$$\mathscr{Y} = \{\mathbf{y} = (y_3, y)|y = (y_1, y_2) \in Y, c^-(y) < y_3 < c^+(y)\}.$$

Here, $y_\alpha = x_\alpha/\varepsilon$ and hence $y_i = x_i/\varepsilon$ or $\mathbf{y} = \mathbf{x}/\varepsilon$. The domain of the plate

$$B_\varepsilon = \{\mathbf{x} = (x_i)|x \in \Omega, \varepsilon c^-(x/\varepsilon) < x_3 < \varepsilon c^+(x/\varepsilon)\}$$

and its boundaries $\Gamma_{\pm}^{\varepsilon}$ and Γ_0^{ε} depend upon ε. For $\varepsilon = \varepsilon_0$, we have $\mathscr{X} = \varepsilon_0\mathscr{Y}$, $Z = \varepsilon_0 Y$, $B = B_{\varepsilon_0}$, $x_3^{\pm}(x) = \varepsilon_0 c^{\pm}(x/\varepsilon_0)$. Note that for every ε, $\mathscr{X}$ and Z are homothetic to $\varepsilon\mathscr{Y}$ and εY, respectively, but B is not homothetic to B_ε if $\varepsilon \neq \varepsilon_0$!

The densities of surface loadings $p_i^{Z\pm}$ are replaced by ε-dependent loadings, as follows:

$$p_\alpha^{Z+} \to p_\alpha^\pm = \varepsilon^2 r_\alpha^\pm(x, x/\varepsilon), \quad p_3^{Z\pm} \to p_3^\pm = \varepsilon^3 q^\pm(x, x/\varepsilon), \tag{6}$$

where $r_\alpha^\pm(x, \cdot)$ and $q^\pm(x, \cdot)$ are Y-periodic functions.

The stress and displacement fields, which solve the (P_ε) problem, are denoted by σ_ε^{ij} and u_i^ε, respectively. The boundary conditions (2) transfer to the form

$$\begin{aligned} \sigma_\varepsilon^{\alpha j}(x, x_3^\pm) n_j^{\pm\varepsilon}(x) &= \varepsilon^2 r_\alpha^\pm(x, x/\varepsilon), \\ \sigma_\varepsilon^{3j}(x, x_3^\pm) n_j^{\pm\varepsilon}(x) &= \varepsilon^3 q^\pm(x, x/\varepsilon), \end{aligned} \tag{7}$$

where $\mathbf{n}_\varepsilon^\pm(x) = (n_j^{\pm\varepsilon}(x))$ represents the unit vector outwardly normal to $\Gamma_\pm^\varepsilon$ at $(x, x_3^\pm)$. If the functions $c^\pm$ can be differentiated, then

$$\mathbf{n}_\varepsilon^\pm(x) = \mathbf{N}^\pm(x/\varepsilon) \cdot (G_\pm(x/\varepsilon))^{-1/2}, \tag{8}$$

where $\mathbf{N}^\pm(\cdot)$, $G_\pm(\cdot)$ are Y-periodic and defined by

$$\begin{aligned} \mathbf{N}^\pm(y) &= [\mp c_{|1}^\pm, \mp c_{|2}^\pm, \pm 1], \\ G_\pm(y) &= 1 + (c_{|1}^\pm)^2 + (c_{|2}^\pm)^2. \end{aligned} \tag{9}$$

The moduli $C_Z^{ijkl}(x, x_3)$ are replaced here by the moduli $\bar{C}^{ijkl}(x/\varepsilon, x_3)$ which are εY-periodic in x. For further convenience, we write

$$\bar{C}^{ijkl}\left(\frac{x}{\varepsilon}, x_3\right) = C^{ijkl}\left(\frac{x_3}{\varepsilon}, \frac{x}{\varepsilon}\right) \tag{10}$$

and $C_Z^{ijkl}(\mathbf{x}) = C^{ijkl}(\mathbf{x}/\varepsilon_0)$. Note that the fields C^{ijkl} are defined on $\mathscr{Y}$ and the functions $C^{ijkl}(\mathbf{y}) = C^{ijkl}(y_3, y)$ are Y-periodic in y.

The original problem (P) is replaced by a family of (P_ε) problems that consist of finding the stress field σ_ε^{ij} such that

$$\int_{B_\varepsilon} \sigma_\varepsilon^{ij} v_{i,j}\, \mathrm{d}\mathbf{x} = \int_{\Gamma_+^\varepsilon} [\varepsilon^2 r_\alpha^+ v_\alpha(x, \varepsilon c^+) + \varepsilon^3 q^+ v_3(x, \varepsilon c^+)]\, \mathrm{d}\Gamma_+^\varepsilon + \int_{\Gamma_-^\varepsilon} [\varepsilon^2 r_\alpha^- v_\alpha(x, \varepsilon c^-) + \varepsilon^3 q^- v_3(x, \varepsilon c^-)]\, \mathrm{d}\Gamma_-^\varepsilon, \tag{11}$$

for every $\mathbf{v}$ vanishing on Γ_0^ε. The elasticity law is assumed as

$$\sigma_\varepsilon^{ij} = C^{ijkl}\left(\frac{x_3}{\varepsilon}, \frac{x}{\varepsilon}\right) \gamma_{kl}(\mathbf{u}^\varepsilon) \tag{12}$$

and $\mathbf{u}^\varepsilon$ vanishes on Γ_0^ε.

In order to make the asymptotic process of constructing solutions to the (P_ε) problem fully deductive, it would be useful to reformulate the (P_ε) problem so that the variable y would play the role of an independent variable. This will be done in the next section.

4. THE VARIATIONAL FORMULATION ON THE PRODUCT DOMAIN $\Omega\times\mathscr{Y}$

Let us define the extrapolation functions as

$$\begin{aligned}
&\Omega\times\mathscr{Y}\ni(x;\mathbf{y})\to\Sigma_\varepsilon^{ij}(x;\mathbf{y})\\
&\Omega\times\mathscr{Y}\ni(x;\mathbf{y})\to U_i^\varepsilon(x;\mathbf{y})\\
&\Omega\times\mathscr{Y}\ni(x;\mathbf{y})\to V_i(x;\mathbf{y}),
\end{aligned}\tag{13}$$

such that they are Y-periodic in y and for $\mathbf{y}=\mathbf{x}/\varepsilon$, they assume values of the stress, displacement and test displacement fields from the (P_ε) problem, i.e.

$$\begin{aligned}
\sigma_\varepsilon^{ij}(x,x_3)&=\Sigma_\varepsilon^{ij}(x;x_3/\varepsilon,x/\varepsilon)\\
u_i^\varepsilon(x,x_3)&=U_i^\varepsilon(x;x_3/\varepsilon,x/\varepsilon)\\
v_i(x,x_3)&=V_i(x;x_3/\varepsilon,x/\varepsilon).
\end{aligned}\tag{14}$$

Further, use will be made of the functions:

$$\begin{aligned}
&B_\varepsilon\times Y\ni(\mathbf{x},y)\to\hat{\Sigma}_\varepsilon^{ij}(\mathbf{x};y)=\Sigma_\varepsilon^{ij}(x;\mathbf{y})\\
&B_\varepsilon\times Y\ni(\mathbf{x},y)\to\hat{U}_i^\varepsilon(\mathbf{x};y)=U_i^\varepsilon(x;\mathbf{y})\\
&B_\varepsilon\times Y\ni(\mathbf{x},y)\to\hat{V}_i(\mathbf{x};y)=V_i(x;\mathbf{y}).
\end{aligned}\tag{15}$$

Note that the functions $r_\alpha^\pm$ and $q^\pm$ are, in fact, functions defined on the product domain $\Omega\times Y$ and hence the extrapolation functions for $r_\alpha^\pm$ and $q^\pm$ are not necessary.

The fields $\hat{U}_i$ and $\hat{V}_i$ are subject to the boundary conditions

$$\hat{U}_i^\varepsilon(\mathbf{x};y)=0,\quad \hat{V}_i(\mathbf{x};y)=0,\quad \mathbf{x}\in\Gamma_0^\varepsilon.\tag{16}$$

The boundary conditions (7) are extrapolated in the form

$$\Sigma_\varepsilon^{\alpha j}\left(x,\frac{x_3^\pm}{\varepsilon},y\right)\cdot n_j^{\pm\varepsilon}(x)=\varepsilon^2\cdot r_\alpha^\pm(x,y)\cdot\left(G_\pm(y)/G_\pm\left(\frac{x}{\varepsilon}\right)\right)^{1/2}\tag{17}$$

$$\Sigma_\varepsilon^{3j}\left(x,\frac{x_3^\pm}{\varepsilon},y\right)\cdot n_j^{\pm\varepsilon}(x)=\varepsilon^3\cdot q_\alpha^\pm(x,y)\cdot\left(G_\pm(y)/G_\pm\left(\frac{x}{\varepsilon}\right)\right)^{1/2}.\tag{18}$$

In order to derive the variational equation that governs the fields Σ_ε and $\mathbf{U}^\varepsilon$, one should start from the equilibrium equations $\sigma_{\varepsilon,j}^{ij}=0$. They can be rewritten in the form

$$\left(\hat{\Sigma}_{\varepsilon,j}^{ij}+\frac{1}{\varepsilon}\cdot\hat{\Sigma}_{\varepsilon|\alpha}^{i\alpha}\right)\Bigg|_{y=x/\varepsilon}=0\tag{19}$$

for $\mathbf{x}\in B_\varepsilon$. Let us extrapolate the above conditions by assuming that

$$\hat{\Sigma}_{\varepsilon,j}^{ij}+\frac{1}{\varepsilon}\hat{\Sigma}_{\varepsilon|\alpha}^{i\alpha}=0\tag{20}$$

holds for arbitrary $\mathbf{x}\in B_\varepsilon$ and $y\in Y$. Hence, in eqn (20) x and y are treated as independent variables, with x_3 and y_3 still being linked by $y_3=x_3/\varepsilon$. Equation (20) encompasses the equilibrium conditions $\sigma_{\varepsilon,j}^{ij}=0$. Upon multiplying both sides of eqn (20) by $\hat{V}_i(\mathbf{x},y)$ and averaging over Y, one obtains

$$\{\hat{\Sigma}^{ij}_{\varepsilon,j}\hat{V}_i\} + \frac{1}{\varepsilon}\{\hat{\Sigma}^{i\alpha}_{\varepsilon|\alpha}\hat{V}_i\} = 0, \tag{21}$$

where $\{\cdot\} = |Y|^{-1}\int_Y(\cdot)\,\mathrm{d}y$ corresponds to averaging over Y, $|Y| = Y_1 Y_2$. In the following, the parentheses $\{\cdot\}$ will only be used to denote the above meaning. By integrating the second term of eqn (21) by parts and taking into account the Y-periodicity of the quantity $\hat{\Sigma}^{i\alpha}\hat{V}_i$, one arrives at

$$\{\hat{\Sigma}^{ij}_{\varepsilon,j}\hat{V}_i\} - \frac{1}{\varepsilon}\{\hat{\Sigma}^{i\alpha}_{\varepsilon}\hat{V}_{i|\alpha}\} = 0. \tag{22}$$

Integrating eqn (22) over B_ε and changing the order of integration, we then have

$$\left\{\int_{B_\varepsilon}\hat{\Sigma}^{ij}_{\varepsilon,j}\hat{V}_i\,\mathrm{d}\mathbf{x}\right\} - \frac{1}{\varepsilon}\left\{\int_{B_\varepsilon}\hat{\Sigma}^{i\alpha}_{\varepsilon}\hat{V}_{i|\alpha}\,\mathrm{d}\mathbf{x}\right\} = 0. \tag{23}$$

On integrating the first term of the above equation by parts and with the help of relations $(15)_{1,3}$ and $\mathrm{d}\Gamma^{\varepsilon}_{\pm} = (G_{\pm}(x/\varepsilon))^{1/2}\,\mathrm{d}x$, one is led to

$$\left\{\int_{B_\varepsilon}\left[\hat{\Sigma}^{i\alpha}_{\varepsilon}\hat{V}_{i,\alpha} + \hat{\Sigma}^{i3}_{\varepsilon}\hat{V}_{i,3} + \frac{1}{\varepsilon}\hat{\Sigma}^{i\alpha}_{\varepsilon}\hat{V}_{i|\alpha}\right]\mathrm{d}\mathbf{x}\right\}$$

$$= \left\{\int_{\Omega}\Sigma^{ij}_{\varepsilon}\left(x,\frac{x_3^+}{\varepsilon};y\right)n_j^{+\varepsilon}(x)\cdot\left(G_+\left(\frac{x}{\varepsilon}\right)\right)^{1/2}\cdot V_i\left(x,\frac{x_3^+}{\varepsilon},y\right)\mathrm{d}x\right\}$$

$$+ \left\{\int_{\Omega}\Sigma^{ij}_{\varepsilon}\left(x,\frac{x_3^-}{\varepsilon};y\right)n_j^{-\varepsilon}(x)\cdot\left(G_-\left(\frac{x}{\varepsilon}\right)\right)^{1/2}\cdot V_i\left(x,\frac{x_3^-}{\varepsilon},y\right)\mathrm{d}x\right\}. \tag{24}$$

Making use of the boundary conditions (17) and (18) and changing the domain of integration from B_ε to $\Omega\times\mathscr{Y}$, one obtains

$$t\varepsilon\int_{\Omega}\langle\Sigma^{i\alpha}_{\varepsilon}V_{i,\alpha} + \frac{1}{\varepsilon}\Sigma^{ij}_{\varepsilon}V_{i|j}\rangle\,\mathrm{d}x = F^{\varepsilon}(\mathbf{V}), \tag{25}$$

where $t = |\mathscr{Y}|/|Y|$ is the average thickness of the cell $\mathscr{Y}$. The linear form $F^{\varepsilon}(\cdot)$ is defined by

$$F^{\varepsilon}(\mathbf{V}) = \int_{\Omega}[\varepsilon^2\{r_\alpha^+(x,y)V_\alpha(x,c^+;y)(G_+(y))^{1/2}\}$$

$$+\varepsilon^2\{r_\alpha^-(x,y)V_\alpha(x,c^-;y)(G_-(y))^{1/2}\} + \varepsilon^3\{q^+(x,y)V_3(x,c^+;y)(G_+(y))^{1/2}\}$$

$$+\varepsilon^3\{q^-(x,y)V_3(x,c^-;y)(G_-(y))^{1/2}\}]\,\mathrm{d}x \tag{26}$$

where the parentheses $\langle\cdot\rangle$ represent averaging over $\mathscr{Y}$, i.e. $\langle\cdot\rangle = |\mathscr{Y}|^{-1}\int_{\mathscr{Y}}(\cdot)\,\mathbf{dy}$; and $|\mathscr{Y}|$ represents the volume of $\mathscr{Y}$.

Now, let $\mathbf{U}$, $\mathbf{V}$ be fields defined on $\Omega\times\mathscr{Y}$ and let $A^{\varepsilon}(\cdot,\cdot)$ be the bilinear form defined by

$$A^{\varepsilon}(\mathbf{U},\mathbf{V}) = t\cdot\varepsilon\int_{\Omega}\langle\Sigma^{i\alpha}V_{i,\alpha}+\frac{1}{\varepsilon}\cdot\Sigma^{ij}V_{i|j}\rangle\,\mathrm{d}x. \tag{27}$$

The extrapolation of Hooke's law (12) is assumed as

$$\Sigma^{ik} = C^{ikmn}(\mathbf{y})\cdot\gamma_{mn}(\mathbf{U}) \tag{28}$$

where

$$\begin{aligned} 2\cdot\gamma_{\alpha\beta}(\mathbf{U}) &= U_{\alpha,\beta}+U_{\beta,\alpha}+\frac{1}{\varepsilon}(U_{\alpha|\beta}+U_{\beta|\alpha}),\\ 2\cdot\gamma_{\alpha 3}(\mathbf{U}) &= U_{3,\alpha}+\frac{1}{\varepsilon}\cdot(U_{\alpha|3}+U_{3|\alpha}),\\ \gamma_{33}(\mathbf{U}) &= \frac{1}{\varepsilon}U_{3|3}. \end{aligned} \tag{29}$$

By virtue of the above notation, the variational equation (25) can be written as

$$A^{\varepsilon}(\mathbf{U}^{\varepsilon},\mathbf{V}) = F^{\varepsilon}(\mathbf{V}) \tag{30}$$

for every $\mathbf{V}\in H(\Omega\times\mathscr{Y})$. The space of kinematically admissible fields of displacements has the form

$$H(\Omega\times\mathscr{Y}) = \{\mathbf{v} = (v_i(x,y))|\mathbf{v}(x,\cdot)\in W(\mathscr{Y}),\mathbf{v}(\cdot,\mathbf{y})\in[H_0^1(\Omega)]^3\},$$

where

$$W(\mathscr{Y}) = \{\mathbf{v}\in[H^1(\mathscr{Y})]^3|\mathbf{v}\text{ assumes equal values at opposite lateral faces of }\mathscr{Y}\}.$$

The field $\mathbf{U}^{\varepsilon}\in H(\Omega\times\mathscr{Y})$ is required to satisfy eqn (30) for every $\mathbf{V}\in H(\Omega\times\mathscr{Y})$. This problem will be called the $(\mathscr{P}_{\varepsilon})$ problem. Having solved this problem, one can find the solution of the (P_{ε}) problem by means of the truncation formulae (14). Thus, the $(\mathscr{P}_{\varepsilon})$ problem encompasses the (P_{ε}) problem and so provides a solution to the original problem (P). The question of well-posedness of the $(\mathscr{P}_{\varepsilon})$ problem will not be addressed in this paper. We only mention here that the $(\mathscr{P}_{\varepsilon})$ problem is not elliptic, which the (P_{ε}) problem is, but parabolic.

5. THE METHOD OF A SMALL PARAMETER

The solution $\mathbf{U}^{\varepsilon} = (U_i^{\varepsilon})$ of the $(\mathscr{P}_{\varepsilon})$ problem is sought in the form used by Caillerie (1984), Kohn and Vogelius (1984) and Kalamkarov *et al.* (1987). However, here y and x are mutually independent:

$$\mathbf{U}^{\varepsilon} = \mathbf{u}^{(0)}(x)+\varepsilon\cdot\mathbf{u}^{(1)}(x;\mathbf{y})+\varepsilon^2\cdot\mathbf{u}^{(2)}(x;\mathbf{y})+\cdots, \tag{31}$$

where $\mathbf{u}^{(0)}\in[H_0^1(\Omega)]^3$ and $\mathbf{u}^{(p)}\in H(\Omega\times\mathscr{Y})$. Consequently, the stresses assume the form of the similar expansion

$$\Sigma_{\varepsilon}^{ij} = \sigma_0^{ij}+\varepsilon\sigma_1^{ij}+\varepsilon^2\sigma_2^{ij}+\cdots, \tag{32}$$

where $\sigma_p^{ij} = \sigma_p^{ij}(x,\mathbf{y})$ and

$$\begin{aligned} \sigma_0^{ij} &= C^{ijkl}u_{k|l}^{(1)}+C^{ijk\beta}u_{k,\beta}^{(0)}\\ \sigma_p^{ij} &= C^{ijkl}u_{k|l}^{(p+1)}+C^{ijk\beta}u_{k,\beta}^{(p)},\ldots\quad p = 1,2,\ldots \end{aligned} \tag{33}$$

Substituting expansion (32) into eqn (30) and equating the terms of the same order with respect to ε results in the reformulation of the $(\mathscr{P}_\varepsilon)$ problem to the following form:

Find $\mathbf{u}^{(0)} \in [H_0^1(\Omega)]^3$ and $\mathbf{u}^{(p)} \in H(\Omega \times \mathscr{Y})$ such that for every $\mathbf{V} \in H(\Omega \times \mathscr{Y})$, the following equations hold true:

$$\int_\Omega \langle \sigma_0^{ij} V_{i|j} \rangle \, \mathrm{d}x = 0, \tag{34}$$

$$\int_\Omega \langle \sigma_0^{i\alpha} V_{i,\alpha} + \sigma_1^{ij} V_{i|j} \rangle \, \mathrm{d}x = 0, \tag{35}$$

$$t \cdot \int_\Omega \langle \sigma_1^{i\alpha} V_{i,\alpha} + \sigma_2^{ij} V_{i|j} \rangle \, \mathrm{d}x = \int_\Omega [\{r_\alpha^+(x,y) V_\alpha(x,c^+,y)(G_+(y))^{1/2}\} + \{r_\alpha^-(x,y) V_\alpha(x,c^-,y)(G_-(y))^{1/2}\}] \, \mathrm{d}x, \tag{36}$$

$$t \int_\Omega \langle \sigma_2^{i\alpha} V_{i,\alpha} + \sigma_3^{ij} V_{i|j} \rangle \, \mathrm{d}x = \int_\Omega [\{q^+(x,y) V_3(x,c^+,y)(G_+(y))^{1/2}\} + \{q^-(x,y) V_3(x,c^-,y)(G_-(y))^{1/2}\}] \, \mathrm{d}x, \tag{37}$$

$$\int_\Omega \langle \sigma_p^{i\alpha} V_{i,\alpha} + \sigma_{p+1}^{ij} \cdot V_{i|j} \rangle \, \mathrm{d}x = 0, \quad p \geqslant 3. \tag{38}$$

The stress fields σ_p^{ij} at $p \geqslant 0$ are interrelated with the displacement fields $\mathbf{u}^{(p)}$ according to the formulae (33).

We shall prove that the first terms of the series (31) can be expressed as

$$u_\alpha^{(0)} = 0, \quad u_3^{(0)} = w(x), \tag{39}$$

$$u_\alpha^{(1)} = v_\alpha(x) - \hat{y}_3 w_{,\alpha}, \quad u_3^{(1)} = v_3(x) \tag{40}$$

$$\begin{aligned} u_\alpha^{(2)} &= \Theta_\alpha^{(\gamma\beta)}(\mathbf{y}) \cdot v_{\gamma,\beta} - \Xi_\alpha^{(\gamma\beta)}(\mathbf{y}) \cdot w_{,\gamma\beta} - \hat{y}_3 v_{3,\alpha} + z_\alpha(x), \\ u_3^{(2)} &= \Theta_3^{(\gamma\beta)}(\mathbf{y}) v_{\gamma,\beta} - \Xi_3^{(\gamma\beta)}(\mathbf{y}) w_{,\gamma\beta} + z_3(x), \end{aligned} \tag{41}$$

$$u_k^{(3)} = \Lambda_k^{(\beta\gamma\delta)}(\mathbf{y}) \cdot v_{\gamma,\delta\beta} - \Xi_k^{(\alpha\beta)}(\mathbf{y}) \cdot v_{3,\alpha\beta} + \Theta_k^{(l\beta)}(\mathbf{y}) z_{l,\beta} - \Pi_k^{(\alpha\gamma\delta)}(\mathbf{y}) \cdot w_{,\gamma\delta\alpha} + u_k^R(x,\mathbf{y}) + \bar{z}_k(x); \tag{42}$$

where $\hat{y}_3 = y_3 - \langle y_3 \rangle$.

The first terms of series (32) for stresses read

$$\sigma_0^{ij} = 0 \tag{43}$$

$$\sigma_1^{ij} = A_0^{ij\alpha\beta}(\mathbf{y}) \cdot v_{\beta,\alpha} - E_0^{ij\alpha\beta}(\mathbf{y}) w_{,\alpha\beta} \tag{44}$$

$$\sigma_2^{ij} = -B_1^{ij\gamma\delta\beta}(\mathbf{y}) \cdot w_{,\gamma\delta\beta} + B_2^{ij\gamma\delta\beta}(\mathbf{y}) \cdot v_{\gamma,\delta\beta} + A_0^{ij\alpha\beta}(\mathbf{y}) z_{\alpha,\beta} - E_0^{ij\alpha\beta}(\mathbf{y}) \cdot v_{3,\alpha\beta} + C^{ijkl}(\mathbf{y}) \cdot u_{k|l}^R. \tag{45}$$

The new quantities involved in eqns (39)–(45) will subsequently be defined below.

Let us define the bilinear form as:

$$a(\mathbf{u}, \mathbf{v}) = \langle C^{ijkl}(\mathbf{y}) u_{i|j} v_{k|l} \rangle, \quad \mathbf{u}, \mathbf{v} \in W(\mathscr{Y}).$$

The functions $\mathbf{\Theta}^{(kl)} \in W(\mathscr{Y})$ are solutions to the variational equation

$$(P^1_{\text{loc}}) \qquad a(\mathbf{\Theta}^{(kl)}, \mathbf{w}) + \langle C^{ijkl} w_{i|j} \rangle = 0 \tag{46}$$

for every $\mathbf{w} \in W(\mathscr{Y})$.

The functions $\mathbf{\Xi}^{(\alpha\beta)} \in W(\mathscr{Y})$ satisfy

$$(P^2_{\text{loc}}) \qquad a(\mathbf{\Xi}^{(\alpha\beta)}, \mathbf{w}) + \langle \hat{y}_3 C^{ij\alpha\beta} w_{i|j} \rangle = 0 \tag{47}$$

for every $\mathbf{w} \in W(\mathscr{Y})$.

The functions $\mathbf{\Lambda}^{(\alpha\gamma\beta)} \in W(\mathscr{Y})$ fulfil the variational equation

$$(P^3_{\text{loc}}) \qquad a(\mathbf{\Lambda}^{(\sigma\gamma\beta)}, \mathbf{w}) + \langle C^{ijk\sigma} \Theta^{(\gamma\beta)}_k w_{i|j} \rangle = 0 \tag{48}$$

for every $\mathbf{w} \in W(\mathscr{Y})$.

The functions $\mathbf{\Pi}^{(\beta\gamma\delta)} \in W(\mathscr{Y})$ satisfy

$$(P^4_{\text{loc}}) \qquad a(\mathbf{\Pi}^{(\beta\gamma\delta)}, \mathbf{w}) + \langle C^{ijk\beta} \Xi^{(\gamma\delta)}_k w_{i|j} \rangle = 0 \tag{49}$$

for every $\mathbf{w} \in W(\mathscr{Y})$.

Let us define the auxiliary functions defined on $\mathscr{Y}$ as

$$A_0^{ijhm} = C^{ijkl} a_{kl}^{(hm)}, \quad E_0^{ij\alpha\beta} = C^{ijkl} e_{kl}^{(\alpha\beta)}, \tag{50}$$

where

$$a_{kl}^{(hm)} = \Theta_{k|l}^{(hm)} + \delta_k^{(h} \delta_l^{m)}, \quad e_{kl}^{(\alpha\beta)} = \Xi_{k|l}^{(\alpha\beta)} + \hat{y}_3 \delta_k^{(\alpha} \delta_l^{\beta)}. \tag{51}$$

The function $\mathbf{u}^R \in H(\Omega \times \mathscr{Y})$ solves the equation

$$(P^5_{\text{loc}}) \qquad a(\mathbf{u}^R, \mathbf{w}) = f(\mathbf{w}) \tag{52}$$

for every $\mathbf{w} \in W(\mathscr{Y})$, where $f(\cdot)$ is a linear form given by

$$f(\mathbf{w}) = \{r_\alpha^+(x,y) w_\alpha(c^+,y)(G_+(y))^{1/2}\}\frac{1}{t} + \{r_\alpha^-(x,y) w_\alpha(c^-,y)(G_-(y))^{1/2}\}\frac{1}{t}$$
$$+ [-\langle E_0^{i\beta\gamma\delta}(\mathbf{y}) w_i(y) \rangle w_{,\beta\gamma\delta} + \langle A_0^{i\beta\gamma\delta}(\mathbf{y}) \cdot w_i(y) \rangle v_{\gamma,\delta\beta}]. \tag{53}$$

In the following, the following tensors are defined on $\mathscr{Y}$

$$\begin{aligned} B_1^{ij\gamma\delta\beta} &= C^{ijk\beta} \Xi_k^{(\gamma\delta)} + C^{ijkl} \Pi_{k|l}^{(\beta\gamma\delta)} \\ B_2^{ij\gamma\delta\beta} &= C^{ijk\beta} \Theta_k^{(\gamma\delta)} + C^{ijkl} \Lambda_{k|l}^{(\beta\gamma\delta)} \end{aligned} \tag{54}$$

and the following averaged tensors

$$\begin{aligned} A_z^{\alpha\beta\lambda\mu} &= \langle A_0^{\alpha\beta\lambda\mu} \rangle, \qquad E_z^{\alpha\beta\lambda\mu} = \langle E_0^{\alpha\beta\lambda\mu} \rangle, \\ F_z^{\alpha\beta\lambda\mu} &= \langle \hat{y}_3 A_0^{\alpha\beta\lambda\mu} \rangle, \quad D_z^{\alpha\beta\lambda\mu} = \langle \hat{y}_3 E_0^{\alpha\beta\lambda\mu} \rangle, \end{aligned} \tag{55}$$

$$B_{z\sigma}^{\lambda\mu\gamma\delta\beta} = \langle B_{\sigma}^{\lambda\mu\gamma\delta\beta}\rangle, \quad D_{z\sigma}^{\lambda\mu\gamma\delta\beta} = \langle \hat{y}_3 B_{\sigma}^{\lambda\mu\gamma\delta\beta}\rangle,$$
$$p^{\lambda\mu} = \langle C^{\lambda\mu kl} u_{k|l}^{R}\rangle, \quad m^{\lambda\mu} = \langle \hat{y}_3 C^{\lambda\mu kl} u_{k|l}^{R}\rangle, \tag{56}$$

will play an important role in the asymptotic process.

The fields $(v_\alpha, w) \in [H_0^1(\Omega)]^2 \times H_0^2(\Omega)$ are solutions to the following boundary value problem

$$(P_{\text{hom}}) \qquad t\int_\Omega \langle \sigma_1^{\alpha\beta}\rangle w_{\alpha,\beta}\,\mathrm{d}x = \int_\Omega r_\alpha(x) w_\alpha(x)\,\mathrm{d} \tag{57}$$

for every $w_\alpha \in H_0^1(\Omega)$;

$$t\int_\Omega [\langle \hat{y}_3 \sigma_1^{\alpha\beta}\rangle \varphi_{\alpha,\beta} + \langle \sigma_2^{\alpha 3}\rangle \varphi_\alpha]\,\mathrm{d}x = \int_\Omega m_\alpha(x)\varphi_\alpha(x)\,\mathrm{d}x \tag{58}$$

for every $\varphi_\alpha \in H_0^1(\Omega)$;

$$t\cdot\int_\Omega \langle \sigma_2^{3\alpha}\rangle v_{,\alpha}\,\mathrm{d}x = \int_\Omega q(x)v(x)\,\mathrm{d}x \tag{59}$$

for every $v \in H_0^1(\Omega)$; where

$$\langle \sigma_1^{\alpha\beta}\rangle = A_z^{\alpha\beta\lambda\mu} v_{\lambda,\mu} - E_z^{\alpha\beta\lambda\mu} w_{,\lambda\mu}$$
$$\langle \hat{y}_3 \sigma_1^{\alpha\beta}\rangle = F_z^{\alpha\beta\lambda\mu} v_{\lambda,\mu} - D_z^{\alpha\beta\lambda\mu} w_{,\lambda\mu}. \tag{60}$$

The quantities $\langle \sigma_2^{3\alpha}\rangle$ can then be eliminated from eqns (58) and (59). The averaged loadings are given by

$$r_\alpha(x) = \{r_\alpha^+(x,y)(G_+(y))^{1/2}\} + \{r_\alpha^-(x,y)(G_-(y))^{1/2}\}$$
$$m_\alpha(x) = \{r_\alpha^+(x,y)\hat{c}^+(y)(G_+(y))^{1/2}\} + \{r_\alpha^-(x,y)\hat{c}^-(y)\cdot(G_-(y))^{1/2}\} \tag{61}$$
$$q(x) = \{q^+(x,y)(G_+(y))^{1/2}\} + \{q^-(x,y)(G_-(y))^{1/2}\},$$

where $\hat{c}^\pm = c^\pm - \langle y_3\rangle$.

Let the function $v_3^0 \in H^2(\Omega)$ fulfil the following non-homogeneous boundary conditions

$$v_3^0 = 0,$$
$$\frac{\partial v_3^0}{\partial \mathbf{n}} = \langle (\hat{y}_3)^2\rangle^{-1}[\langle \hat{y}_3 \Theta_\alpha^{(\gamma\beta)}\rangle n_\alpha v_{\gamma,\beta} - \langle \hat{y}_3 \Xi_\alpha^{(\gamma\beta)}\rangle n_\alpha w_{,\gamma\beta}]$$

along γ.

Having found the fields $(\mathbf{v}, w)$, one can determine the functions $\mathbf{z} = (z_\alpha) \in [H_0^1(\Omega)]^2$ and v_3 [such that $v_3 - v_3^0 \in H_0^2(\Omega)$] as the solution to the following boundary value problem

$$(P'_{\text{hom}}) \qquad \int_\Omega \langle \sigma_2^{\lambda\mu}\rangle w_{\lambda,\mu}\,\mathrm{d}x = 0$$
$$\int_\Omega \langle \hat{y}_3 \sigma_2^{\lambda\mu}\rangle v_{,\lambda\mu}\,\mathrm{d}x = 0 \tag{62}$$

for every $\mathbf{w} \in [H_0^1(\Omega)]^2$ and $v \in H_0^2(\Omega)$; where

$$\langle \sigma_2^{\lambda\mu} \rangle = A_z^{\lambda\mu\alpha\beta} z_{\alpha,\beta} - E_z^{\lambda\mu\alpha\beta} v_{3,\alpha\beta} + \sigma_{20}^{\lambda\mu}(x)$$
$$\langle \hat{y}_3 \sigma_2^{\lambda\mu} \rangle = F_z^{\lambda\mu\alpha\beta} z_{\alpha,\beta} - D_z^{\lambda\mu\alpha\beta} v_{3,\alpha\beta} + m_{20}^{\lambda\mu}(x). \tag{63}$$

The quantities $\boldsymbol{\sigma}_{20}$ and $\mathbf{m}_{20}$ depend upon the fields $\mathbf{v}$ and w, according to the formulae

$$\sigma_{20}^{\lambda\mu} = -B_{z1}^{\lambda\mu\gamma\delta\beta} w_{,\gamma\delta\beta} + B_{z2}^{\lambda\mu\gamma\delta\beta} v_{\gamma,\delta\beta} + p^{\lambda\mu}$$
$$m_{20}^{\lambda\mu} = -D_{z1}^{\lambda\mu\gamma\delta\beta} w_{,\gamma\delta\beta} + D_{z2}^{\lambda\mu\gamma\delta\beta} v_{\gamma,\delta\beta} + m^{\lambda\mu}, \tag{64}$$

where the tensors $p^{\lambda\mu}$ and $m^{\lambda\mu}$ have been defined by eqns $(56)_{3,4}$.

Still undetermined, the fields z_3 and $\bar{z}_k$ remain. They can be found in the subsequent step of the asymptotic process. Note that these fields do not affect the distribution of the stress fields σ_1^{ij} and σ_2^{ij}.

A rather lengthy derivation of the formulae (39)–(64) will be divided below into a series of subsequent steps. The derivation of the formulae (39)–(41), (43), (44), (46), (47), (60) and (61) has been outlined in the paper by Kalamkarov *et al.* (1987). In some special cases, derivations have been reported in the works of Caillerie (1984) and Kohn and Vogelius (1984). However, in order to make the present paper legible and complete, it was thought useful to derive all the formulae (39)–(64).

Step 1

Substituting eqn (33) into eqn (34) and choosing $V_i = \varphi(x) \cdot w_i(\mathbf{y})$, $\varphi \in \mathscr{D}(\Omega)$ and $\mathbf{w} \in W(\mathscr{Y})$ and, due to the function φ being arbitrary, the equation that governs the function $\mathbf{u}^{(1)}$ can be arrived at as

$$\langle [C^{ijkl} u_{k|l}^{(1)} + C^{ij\beta k} u_{k,\beta}^{(0)}] w_{i|j} \rangle = 0. \tag{65}$$

The solution can be expressed as

$$u_k^{(1)} = \Theta_k^{(j\beta)}(\mathbf{y}) u_{j,\beta}^0 + v_k(x), \tag{66}$$

where the functions $\boldsymbol{\Theta}^{(j\beta)} = (\Theta_k^{(j\beta)}) \in W(\mathscr{Y})$ are solutions to the $(P_{\rm loc}^1)$ problem $(k = j, l = \beta)$. In the case of constant plate thickness, the $(P_{\rm loc}^1)$ problem is well-posed; its solution exists and is determined up to an additive constant (cf. Caillerie, 1984). However, the same arguments concerning existence and uniqueness transfer to the considered case of varying thickness. In order to make the solution unique, we demand that it has a vanishing mean value over $\mathscr{Y}$

$$\langle \boldsymbol{\Theta}^{(kh)} \rangle = 0. \tag{67}$$

Note that the function

$$\boldsymbol{\Theta}^{(3\beta)} = (\Theta_k^{(3\beta)}) = (-y_3 \delta_k^\beta) \tag{68}$$

satisfies

$$C^{ijkh} \Theta_{k|h}^{(3\beta)} + C^{ij3\beta} = A_0^{ij3\beta} = 0, \tag{69}$$

and hence function (68) solves the $(P_{\rm loc}^1)$ problem. The solution which satisfies condition (67) has the form

$$\Theta_k^{(3\beta)} = -\hat{y}_3 \cdot \delta_k^\beta. \tag{70}$$

For plates which are symmetric with respect to the plane $x_3 = 0$, $\hat{y}_3 = y_3$ holds.

Let us fix (k, l) in the $(P_{\rm loc}^1)$ problem. The field $\boldsymbol{\Theta}^{(kl)} = \boldsymbol{\Theta}^{(lk)}$ is associated with the state of initial deformation

$$\boldsymbol{\varepsilon}_0^{(kl)} = [\tfrac{1}{2}(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l)], \tag{71}$$

which is

$$\boldsymbol{\varepsilon}_0^{(11)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_0^{(12)} = \boldsymbol{\varepsilon}_0^{(21)} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_0^{(22)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The states (71) are linearly independent. Thus, the deformation states $\mathbf{a}^{(kl)}$ [eqn $(51)_1$] are also linearly independent since the (P^1_{loc}) problem is linear.

On substituting eqn (70) into eqn (66) one obtains

$$\begin{aligned} u_\sigma^{(1)} &= \Theta_\sigma^{(\alpha\beta)}(\mathbf{y})\cdot u^{(0)}_{\alpha,\beta} - \hat{y}_3 w_{,\sigma} + v_\sigma(x), \\ u_3^{(1)} &= \Theta_3^{(\alpha\beta)}(\mathbf{y})\cdot u^{(0)}_{\alpha,\beta} + v_3(x). \end{aligned} \tag{72}$$

Step 2

Substitute relations (72) into eqns $(33)_1$, but remembering (69), we obtain

$$\sigma_0^{ij} = A_0^{ij\alpha\beta} u^{(0)}_{\alpha,\beta}, \tag{73}$$

where $\mathbf{A}_0$ has been defined by $(50)_1$. By substituting relations (73) into (35) and putting $V_\alpha = w_\alpha(x)$, $V_3 = 0$ and $w_\alpha \in H_0^1(\Omega)$, we find that

$$\int_\Omega \langle \sigma_0^{\alpha\beta}\rangle w_{\beta,\alpha}\,\mathrm{d}x = 0 \quad \text{for} \quad \mathbf{w}\in [H_0^1(\Omega)]^2; \tag{74}$$

where

$$\langle \sigma_0^{\alpha\beta}\rangle = A_z^{\alpha\beta\gamma\delta} u^{(0)}_{\gamma,\delta}. \tag{75}$$

The tensor $\mathbf{A}_z$ has been defined by $(55)_1$. This tensor fulfils the symmetry conditions

$$A_z^{\alpha\beta\lambda\mu} = A_z^{\lambda\mu\alpha\beta}, \quad A_z^{\alpha\beta\lambda\mu} = A_z^{\beta\alpha\lambda\mu} = A_z^{\beta\alpha\mu\lambda} \tag{76}$$

and is positively determined. To prove (76), let us fix $k = \alpha$ and $l = \beta$ in eqn (46) and put $w_i = \Theta_i^{(\lambda\mu)}$. Thus, we get an identity which when added to both sides of eqn $(55)_1$ results in an alternative definition of $\mathbf{A}_z$,

$$A_z^{\alpha\beta\lambda\mu} = \langle C^{ijkl} a_{ij}^{(\alpha\beta)} a_{kl}^{(\lambda\mu)}\rangle, \tag{77}$$

which readily fulfils the symmetry relations of (76). By virtue of the positive definiteness of the elasticity tensor $\mathbf{C}$, we can estimate

$$\begin{aligned} \sigma(\gamma^0) &= A_z^{\alpha\beta\lambda\mu}\gamma^0_{\alpha\beta}\gamma^0_{\lambda\mu} \geqslant c_1\langle a_{ij}^{(\alpha\beta)} a_{ij}^{(\lambda\mu)}\rangle \gamma^0_{\alpha\beta}\gamma^0_{\lambda\mu} \\ &= c_1\langle \bar{\gamma}^0_{ij}\bar{\gamma}^0_{ij}\rangle, \quad c_1 = \text{const} > 0, \end{aligned} \tag{78}$$

where $\bar{\gamma}^0_{ij} = a_{ij}^{(\alpha\beta)}\gamma^0_{\alpha\beta}$, $(\gamma^0_{\alpha\beta}) = (\gamma^0_{\beta\alpha}) = \gamma^0$. Thus, the form $\sigma(\cdot)$ is non-negative. We shall prove that $\sigma(\gamma^0) = 0$ implies $\gamma^0 = 0$. The equality $\sigma(\gamma^0) = 0$ leads to $\bar{\gamma}^0_{ij} = 0$. Since the fields $a^{(\alpha\beta)}$ are linearly independent (cf. Step 1), $\gamma^0_{\alpha\beta} = 0$. Thus, the tensor $\mathbf{A}_z$ is positively definite and we conclude that the problem (74)–(75) is elliptic and its solution is trivial: $u_\alpha^{(0)} = 0$. Thus, we infer that $\sigma_0^{ij} = 0$ [eqn (43)].

Step 3

On substituting the formulae (72) into eqn (33) for $p = 1$, whilst remembering that $\mathbf{u}^{(0)} = 0$, one obtains

$$\sigma_1^{ij} = C^{ijkl}u_{k|l}^{(2)} + C^{ij\alpha k}v_{k,\alpha} - \hat{y}_3 C^{ij\alpha\beta}w_{,\alpha\beta}. \tag{79}$$

By inserting the above relation into eqn (35), considering (43), taking the test functions as $V_i = \varphi(x)\cdot w_i(\mathbf{y})$, $\varphi \in \mathscr{D}(\Omega)$ and $\mathbf{w} \in W(\mathscr{Y})$, and making use of the arbitrariness of the functions φ, we arrive at the local equation

$$\langle [C^{ijkl}u_{k|l}^{(2)} + C^{ij\beta k}v_{k,\beta} - \hat{y}_3 C^{ij\alpha\beta}w_{,\alpha\beta}]w_{i|j}\rangle = 0 \tag{80}$$

for every $\mathbf{w} \in W(\mathscr{Y})$. The solution to this can be expressed as

$$\mathbf{u}^{(2)} = \mathbf{\Theta}^{(j\beta)}(\mathbf{y})\cdot v_{j,\beta} - \mathbf{\Xi}^{(\alpha\beta)}(\mathbf{y})\cdot w_{,\alpha\beta} + \mathbf{z}(x), \tag{81}$$

where $\mathbf{z} = (z_\alpha)$ are at this moment undetermined, $\mathbf{\Theta}^{(j\beta)}$ are solutions to the (P_{loc}^1) problem, subject to condition (67), and the functions $\mathbf{\Xi}^{(\alpha\beta)} = \mathbf{\Xi}^{(\beta\alpha)} \in W(\mathscr{Y})$ are solutions to the (P_{loc}^2) problem. They are also normalized as

$$\langle \mathbf{\Xi}^{(\alpha\beta)}\rangle = 0. \tag{82}$$

The (P_{loc}^2) problem, augmented with the above condition, is well-posed. Its solution exists and is unique. In the case of constant plate thickness, the proof of well-posedness has been given by Caillerie (1984). The proof in the more general case considered is similar and will not be reported. Just as the fields $\mathbf{\Theta}^{(kl)}$, $\mathbf{\Xi}^{(\alpha\beta)}$ are linearly independent, hence also the fields $\mathbf{e}^{(\alpha\beta)}$, defined by eqn $(51)_2$ have the same property. Moreover, the fields $\mathbf{a}^{(kl)}$ and $\mathbf{e}^{(\alpha\beta)}$ are mutually linearly independent. This fact will play an important role in the course of the proof that the (P_{hom}) problem is well established.

Substituting the relation (70) into eqn (81) results in eqns (41). Upon inserting (81) into (79), one arrives at eqn (44) which involves the tensor $\mathbf{E}_0$, previously defined by $(50)_2$. Note that the field v_3 contributes to $u_3^{(1)}$ but does not affect the field (σ_1^{ij}).

Step 4

If $V_3 = 0$, $V_\alpha = w_\alpha(x)$ and $w_\alpha \in H_0^1(\Omega)$ in eqn (35), then $V_\alpha(x, c^\pm, y) = w_\alpha(x)$ and hence eqn (57) occurs. The constitutive relationship for $\langle \sigma_1^{\alpha\beta}\rangle$ is then inferred from eqn (44) and assumes the form $(60)_1$.

Step 5

If $V_3 = 0$, $V_\alpha = \hat{y}_3\cdot\varphi_\alpha(x)$ and $\varphi_\alpha \in H_0^1(\Omega)$ in eqn (36), then $V_\alpha(x, c^\pm(y), y) = \hat{c}^\pm(y)\cdot\varphi_\alpha(x)$ and we obtain eqn (58). The constitutive equation for $\langle \hat{y}_3\sigma_1^{\alpha\beta}\rangle$ assumes the form $(60)_2$.

Step 6

By putting $V_\alpha = 0$ and $V_3 = v(x)$ and $v \in H_0^1(\Omega)$ into eqn (37), eqn (59) occurs.

Step 7

We shall prove the symmetries

$$E_z^{\alpha\beta\lambda\mu} = F_z^{\lambda\mu\alpha\beta}, \quad D_z^{\alpha\beta\lambda\mu} = D_z^{\lambda\mu\alpha\beta} \tag{83}$$

and the positive definiteness of the tensor $\mathbf{D}_z$. If we set $w_i = \Xi_i^{(\lambda\mu)}$ in (P_{loc}^1) $(k = \alpha,\ l = \beta)$ and $w_i = \Theta_i^{(\alpha\beta)}$ in (P_{loc}^2) $(\alpha = \lambda,\ \beta = \mu)$, then upon subtracting the identities obtained one obtains

$$\langle C^{ij\alpha\beta}\Xi^{(\lambda\mu)}_{i|j}\rangle = \langle \hat{y}_3 C^{ij\lambda\mu}\Theta^{(\alpha\beta)}_{i|j}\rangle. \tag{84}$$

The above identity implies condition $(83)_1$. Moreover, one can show that

$$E_z^{\alpha\beta\lambda\mu} = \langle C^{ijkl}a^{(\alpha\beta)}_{ij}e^{(\lambda\mu)}_{kl}\rangle,$$
$$F_z^{\alpha\beta\lambda\mu} = \langle C^{ijkl}e^{(\alpha\beta)}_{ij}a^{(\lambda\mu)}_{kl}\rangle,$$
$$D_z^{\alpha\beta\lambda\mu} = \langle C^{ijkl}e^{(\alpha\beta)}_{ij}e^{(\lambda\mu)}_{kl}\rangle, \tag{85}$$

where $e^{(\alpha\beta)}_{ij}$ have been defined by $(51)_2$. The last formula satisfies condition $(83)_2$ identically. The same formula makes it possible to prove that the tensor $\mathbf{D}_z$ is positively definite. The proof is similar to the proof that the tensor $\mathbf{A}_z$ is positively definite, cf. Step 2. Like $a^{(\alpha\beta)}_{ij}$, similarly $e^{(\alpha\beta)}_{ij}$ are linearly independent (Step 3), which is why the proof outlined in Step 2 transfers to the case considered here.

Now we are able to prove that the homogenized problem (P_{hom}) is well-posed. The elastic potential of the effective plate reads

$$2j(\boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h) = A_z^{\alpha\beta\lambda\mu}\gamma^h_{\alpha\beta}\gamma^h_{\lambda\mu} + D_z^{\alpha\beta\lambda\mu}\kappa^h_{\alpha\beta}\kappa^h_{\lambda\mu} + E_z^{\alpha\beta\lambda\mu}\gamma^h_{\alpha\beta}\kappa^h_{\lambda\mu} + F_z^{\alpha\beta\lambda\mu}\kappa^h_{\alpha\beta}\gamma^h_{\lambda\mu}, \tag{86}$$

where

$$\gamma^h_{\alpha\beta} = \tfrac{1}{2}\cdot(v_{\alpha,\beta} + v_{\beta,\alpha}), \quad \kappa^h_{\alpha\beta} = -w_{,\alpha\beta}.$$

By virtue of identities (84), (85) and (77) and definitions (55), one can rearrange the potential j to have the form

$$2j(\boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h) = \langle C^{ijkl}\bar{\gamma}_{ij}\bar{\gamma}_{kl}\rangle, \tag{87}$$

where

$$\bar{\gamma}_{ij} = a^{(\lambda\mu)}_{ij}(\mathbf{y})\gamma^h_{\lambda\mu} + e^{(\lambda\mu)}_{ij}(\mathbf{y})\cdot\kappa^h_{\lambda\mu}.$$

Due to the positive definiteness of the elasticity tensor $\mathbf{C}$, one can estimate

$$j(\boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h) \geqslant \text{const.}\ \langle\bar{\gamma}_{ij}\bar{\gamma}_{ij}\rangle, \tag{88}$$

which proves that j is non-negative. We shall prove that when $j = 0$, this implies that $\gamma^h_{\lambda\mu} = 0$ and $\kappa^h_{\lambda\mu} = 0$. Let $j = 0$. Then $\bar{\gamma}_{ij} = 0$. In the Step 3 we have already proved that the states $\mathbf{a}^{(\alpha\beta)}$ and $\mathbf{e}^{(\alpha\beta)}$ are linearly independent. Hence, the equality $\bar{\gamma}_{ij} = 0$ implies that $\boldsymbol{\gamma}^h$ and $\boldsymbol{\kappa}^h$ vanish. Thus, there exists a positive constant c, such that

$$j(\boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h) \geqslant c\cdot(\gamma^h_{\alpha\beta}\gamma^h_{\alpha\beta} + \kappa^h_{\alpha\beta}\kappa^h_{\alpha\beta}) \tag{89}$$

for every $\gamma^h_{\alpha\beta} = \gamma^h_{\alpha\beta}$ and $\kappa^h_{\alpha\beta} = \kappa^h_{\alpha\beta}$. Now, according to standard arguments and as proved in a paper by Caillerie (1984), one can show that the solution $(\mathbf{v}, w)$ exists and is unique.

Step 8

We shall prove the identities (cf., Kalamkarov *et al.*, 1987)

$$\langle\sigma^{3j}_1\rangle = 0, \quad \langle\hat{y}_3\sigma^{3j}_1\rangle = 0. \tag{90}$$

To this end let us substitute $w_i = \delta_{im}\hat{y}_3$ in (P^1_{loc}) and hence

$$\langle A_0^{m3kl}\rangle = 0. \tag{91}$$

If we now substitute $w_i = \frac{1}{2}\delta_{im}(\hat{y}_3)^2$ in (P^1_{loc}), then

$$\langle F_0^{m3kl} \rangle = 0. \tag{92}$$

If $w_i = \delta_{im} \cdot \hat{y}_3$ in (P^2_{loc}), one finds

$$\langle E_0^{m3\alpha\beta} \rangle = 0. \tag{93}$$

On the other hand, the substitution of $w_i = \frac{1}{2}(\hat{y}_3)^2 \cdot \delta_{im}$ into (P^2_{loc}) results in

$$\langle D_0^{m3\alpha\beta} \rangle = 0. \tag{94}$$

Thus, equalities (90) hold true. In virtue of equality $(90)_1$, substitution of $V_3 = v \in H_0^1(\Omega)$, $V_\alpha = 0$ into eqn (36) leads to an equality that is satisfied identically.

Step 9

Insert the formula (41) into eqn (33) with $p = 2$, and one obtains

$$\sigma_2^{ij} = C^{ijkl} u_{k|l}^{(3)} + C^{ijk\sigma} \Theta_k^{(\gamma\beta)} v_{\gamma,\beta\sigma} - C^{ijk\sigma} \Xi_k^{(\gamma\beta)} w_{,\gamma\beta\sigma} - \hat{y}_3 C^{ij\alpha\sigma} v_{3,\alpha\sigma} + C^{ijk\sigma} z_{k,\sigma}. \tag{95}$$

Substituting $V_i = w_i(\mathbf{y})\varphi(x)$, $\varphi \in \mathscr{D}(\Omega)$ and $\mathbf{w} \in W(\mathscr{Y})$ in eqn (36), integrating by parts and localizing, one arrives at

$$-\langle \sigma_1^{i\alpha} w_i \rangle_{,\alpha} + \langle \sigma_2^{ij} w_{i|j} \rangle = \frac{1}{t} [\{ r_\alpha^+(x,y) w_\alpha(c^+,y)(G_+(y))^{1/2} \} + \{ r_\alpha^-(x,y) w_\alpha(c^-,y)(G_-(y))^{1/2} \}] \tag{96}$$

for every $\mathbf{w} \in W(\mathscr{Y})$. Now, with the help of relations (95) and (44), the equation for field $u_k^{(3)}$ can be found as

$$a(\mathbf{u}^{(3)}, \mathbf{w}) = -\langle C^{ijk\sigma} \Theta_k^{(\gamma\beta)} w_{i|j} \rangle v_{\gamma,\beta\sigma} + \langle C^{ijk\sigma} \Xi_k^{(\gamma\beta)} w_{i|j} \rangle w_{,\gamma\beta\sigma} + \langle C^{ij\alpha\sigma} \hat{y}_3 w_{i|j} \rangle v_{3,\alpha\sigma} - \langle C^{ijk\sigma} w_{i|j} \rangle z_{k,\sigma} + \langle \sigma_1^{i\alpha} w_i \rangle_{,\alpha} + \frac{1}{t} [\{ r_\alpha^+(x,y) w_\alpha(c^+,y)(G_+(y))^{1/2} \} + \{ r_\alpha^-(x,y) w_\alpha(c^-,y)(G_-(y))^{1/2} \}] \tag{97}$$

for every $\mathbf{w} \in W(\mathscr{Y})$. Note that $\sigma_1^{i\alpha}$ depend on (v_α, w).

The solution of the above equation can be expressed as

$$u_k^{(3)} = \Lambda_k^{(\beta\gamma\delta)}(\mathbf{y}) v_{\gamma,\delta\beta} - \Xi_k^{(\alpha\beta)}(\mathbf{y}) v_{3,\alpha\beta} - \Pi_k^{(\beta\gamma\delta)}(\mathbf{y}) \cdot w_{,\beta\gamma\delta} + \Theta_k^{(l\beta)}(\mathbf{y}) z_{l,\beta} + u_k^R + \bar{z}_k(x), \tag{98}$$

where the fields $\Lambda_k^{(\beta\gamma\delta)}$ and $\Pi_k^{(\beta\gamma\delta)}$ satisfy eqns (48) and (49). For sufficiently regular data, the (P^3_{loc}) and (P^4_{loc}) problems possess unique solutions; the relevant regularity problem will not be discussed in this paper. Formula (98) involves the field $\mathbf{u}^R$ which depends on (v_α, w); this field fulfils the following variational equation

$$a(\mathbf{u}^R, \mathbf{w}) = \langle \sigma_1^{i\alpha} w_i \rangle_{,\alpha} + \frac{1}{t} [\{ r_\alpha^+(x,y) w_\alpha(c^+,y)[G_+(y)]^{1/2} \} + \{ r_\alpha^-(x,y) w_\alpha(c^-,y)[G_-(y)]^{1/2} \}] \tag{99}$$

for every $\mathbf{w} \in W(\mathscr{Y})$. Substitution of relation (44) into (99) produces eqn (52). For sufficiently regular data, the linear form (53) is continuous. Moreover, this form vanishes on $\mathbf{w} = \text{const}$ since, by virtue of eqns (90) and (57), the right-hand side of eqn (99) vanishes for constant fields. Thus, the (P^5_{loc}) problem is well stated, cf. one of the theorems of Caillerie (1984). The fields $\bar{z}_k$ involved in eqn (98) can be found in the next step of the procedure.

Finally, formula (42) has been substantiated. With the help of relation (98), the fields σ_2^{ij} can be expressed in terms of the fields w, (v_α) and v_3, (z_α). Substitution of relation (98) into eqn (95) results in relation (45). By virtue of the identity (69), the z_3 field does not contribute to the expression for σ_2^{ij}.

Step 10

We shall now find the equations for the fields v_3, (z_α). To this end, we insert $V_3 = 0$ and $V_\alpha = w_\alpha(x)$ into eqn (37) and arrive at eqn $(62)_1$. In eqn (37), we then assume that $V_\alpha = \hat{y}_3 w_\alpha(x)$, $w_\alpha = v_{,\alpha}$, $v \in H_0^2(\Omega)$ and $V_3 = 0$. In this way we obtain the following:

$$\int_\Omega [\langle \hat{y}_3 \sigma_2^{\lambda\mu} \rangle v_{,\lambda\mu} + \langle \sigma_3^{3\alpha} \rangle v_{,\alpha}]\, dx = 0. \tag{100}$$

Substituting $V_\alpha = 0$ and $V_3 = v(x) \in H_0^2(\Omega)$ into eqn (38) with $p = 3$, gives

$$\int_\Omega \langle \sigma_3^{3\alpha} \rangle v_{,\alpha}\, dx = 0. \tag{101}$$

Thus, eqn $(62)_2$ holds true. Equations (62) are governing variational equations for the fields (z_α), v_3. Relationships (63) follow from (45) and conditions $(16)_1$ imply that

$$\mathbf{u}^{(k)}(\mathbf{x}, y) = \mathbf{0}, \quad \mathbf{x} \in \Gamma_0^\varepsilon, \quad k \geqslant 0. \tag{102}$$

The assumption that $V_3 = 0$ on γ assures the above condition to be fulfilled for $k = 1$. Conditions (102) will be satisfied for $k = 2$ provided that

$$\begin{aligned} \Theta_\alpha^{(\gamma\beta)}(\mathbf{y}) v_{\gamma,\beta} - \Xi_\alpha^{(\gamma\beta)}(\mathbf{y}) w_{,\gamma\beta} - \hat{y}_3 \cdot v_{3,\alpha} + z_\alpha &= 0, \\ \Theta_3^{(\gamma\beta)}(\mathbf{y}) v_{\gamma,\beta} - \Xi_3^{(\gamma\beta)}(\mathbf{y}) w_{,\gamma\beta} + z_3(x) &= 0 \end{aligned} \tag{103}$$

for $\mathbf{x} \in \Gamma_0^\varepsilon$. The above conditions cannot be identically satisfied and thus, it was considered reasonable to replace conditions $(103)_1$ by their averages with weights 1 and $\hat{y}_3$ over $\mathscr{Y}$, and replace condition $(103)_2$ by the average over $\mathscr{Y}$. By virtue of (67) and (82) and of the equality $\langle \hat{y}_3 \rangle = 0$, which follows from the very definition of $\hat{y}_3$, the boundary condition $\langle 1 \cdot \mathbf{u}^{(2)} \rangle = \mathbf{0}$ reduces to $\langle z_k \rangle = z_k = 0$ along γ. Similarly, the conditions $\langle \hat{y}_3 u_\alpha^{(2)} \rangle = 0$ result in a reduced condition of the form

$$\langle (\hat{y}_3)^2 \rangle v_{3,\alpha} = \langle \hat{y}_3 \Theta_\alpha^{(\gamma\beta)}(\mathbf{y}) \rangle v_{\gamma,\beta} - \langle \hat{y}_3 \Xi_\alpha^{(\gamma\beta)}(\mathbf{y}) \rangle w_{,\gamma\beta}. \tag{104}$$

Upon multiplying the above relation by τ_α (τ_α being components of the versor tangent to γ), one obtains the values that should be reached by the derivative $\partial v_3/\partial s$. On the other hand, $v_3 = 0$ on γ, hence $\partial v_3/\partial s = 0$. This contradiction discloses that assumption (31) is deficient in appropriate boundary layer terms. The condition $v_3 = 0$ on γ entails the vanishing of $u_3^{(1)}$, a term of the order $O(\varepsilon)$, and hence more important than the term $u_\alpha^{(2)}$. Thus, we retain the condition $v_3 = 0$ on γ and discard the condition concerning $\partial v_3/\partial s$ that follows from eqn (104). However, eqn (104) determines the values of $\partial v_3/\partial \mathbf{n}$ along γ, which is not in conflict with the previous boundary conditions. One can find a function $v_3^0 \in H^2(\Omega)$ that vanishes on γ such that its derivative $\partial v_3^0/\partial \mathbf{n} = v_{3,\alpha}^0 n_\alpha$ is determined by the right-hand side of eqn (104). This function helps us formulate the (P'_{hom}) problem for the functions $((v_\alpha), w)$ and thus, the (P'_{hom}) problem is eventually established. The well-posedness of this problem is assured according to arguments similar to those that assure the (P_{hom}) problem to be correctly posed.

6. FINAL REMARKS

The asymptotic expansion (31) is too simple to satisfy exactly the kinematical boundary condition on the clamped edge of the plate. The discrepancies have been moderated by substituting the averaged conditions $\langle u_k^{(2)} \rangle = 0$ and $\langle \hat{y}_3 u_\alpha^{(2)} n_\alpha \rangle = 0$ for the condition $u_k^{(2)} = 0$. By virtue of these assumptions, the subsequent higher-order homogenized problem (P'_{hom}) could have been put forward.

How to generalize the "ansatz" (31) so that it would properly comprise the boundary layer effects seems to be an open problem. These new terms are likely to be much more difficult to express than those in the theory of homogeneous plates, although the latter are not quite so easy to find (cf. Pécastaings, 1985).

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